POINCARÉ FUNCTIONS WITH SPIDERS' WEBS

HELENA MIHALJEVIĆ-BRANDT AND JÖRN PETER

Abstract. For a polynomial p with a repelling fixed point z_0, we consider *Poincaré functions* of p at z_0, i.e. entire functions L which satisfy $L(0) = z_0$ and $p(L(z)) = L(p'(z_0) \cdot z)$ for all $z \in \mathbb{C}$. We show that if the component of the Julia set of p that contains z_0 equals $\{z_0\}$, then the (fast) escaping set of L is a *spider's web*; in particular it is connected. More precisely, we classify all linearizers of polynomials with regards to the spider's web structure of the set of all points which escape faster than the iterates of the maximum modulus function at a sufficiently large point R.

1. Introduction

Let f be a transcendental entire function. With the fundamental work of Eremenko [4], the *escaping set*

$$\mathrm{I}(f) := \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

has become an intensively studied object in transcendental holomorphic dynamics. Since then, much progress has been achieved in exploring the topological and dynamical properties of the escaping set and some of its subsets (for some results, see [9, 13, 14, 15, 16, 17]).

Rippon and Stallard discovered that the *fast escaping set* $\mathrm{A}(f)$, which was originally introduced by Bergweiler and Hinkkanen [2], shares many significant features with $\mathrm{I}(f)$. If we set $M(f,r) := \max_{|z|=r} |f(z)|$ and choose any constant R such that

$$\mathrm{M}(f,r) > r \text{ whenever } r \geq R, \tag{1.1}$$

the fast escaping set of f can be described as

$$\mathrm{A}(f) = \bigcup_{l \in \mathbb{N}} \mathrm{A}_R^{-l}(f),$$

where $\mathrm{A}_R^l(f)$ are the so-called *level sets*, defined by

$$\mathrm{A}_R^l(f) := \{z \in \mathbb{C} : |f^{n+l}(z)| \geq \mathrm{M}^n(R), n \geq \max\{0, -l\}\}.$$

(Throughout the article M^n denotes the n-th iterate of the maximum modulus function.)

Recently, Rippon and Stallard [16, 14] introduced the concept of an *(infinite) spider's web*. This is a connected set $E \subset \mathbb{C}$ with the property

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that there exists a sequence of increasing simply-connected domains (G_n) whose union is all of $\mathbb{C}$ such that $\partial G_n \subset E$ for all n. Functions whose (fast) escaping set is a spider's web have some strong dynamical properties. For instance, every such function has only bounded Fatou components and there exists no curve to ∞ on which f is bounded (compare [16]). In particular, the set of singular values of f must be unbounded. (For precise definitions see Section 2).

In [16], various sufficient criteria are presented such that $\mathrm{I}(f)$ and $\mathrm{A}(f)$ is a spider's web. Primarily, this is the case whenever the set

$$\mathrm{A}_R(f) := \mathrm{A}_R^0(f)$$

is a spider's web for any R as in (1.1).

In this paper, we present a large and interesting class of functions whose escaping set is a spider's web, namely, Poincaré functions of certain polynomials. To make this precise, let p be a polynomial with a repelling fixed point z_0 (i.e. $p(z_0) = z_0$ and $|p'(z_0)| > 1$). Then there exists an entire function L called a *Poincaré function* or a *linearizer of p at z_0* which satisfies

$$L(0) = z_0 \quad \text{and} \quad p(L(z)) = L(p'(z_0) \cdot z) \text{ for all } z \in \mathbb{C}.$$

In the above functional equation, we can iterate the function p; this already indicates that the analysis of a linearizer strongly depends on the dynamical properties of p. However, L does not only depend on p but also on z_0 and $p'(z_0)$ which makes linearizers good candidates for constructing functions with various interesting analytical properties (see e.g. Section 3). Furthermore, they are naturally good candidates for constructing gauge functions to estimate the Hausdorff measure of escaping and Julia sets of exponential functions (see [11]).

It was conjectured by Rempe that the escaping set of a linearizer of a quadratic polynomial for which the critical point escapes is a spider's web. In this article, we show that this is true; moreover, we classifiy all linearizers of polynomials corresponding to whether the sets $\mathrm{A}_R(L)$ are spiders' webs or not.

Theorem 1.1. *Let p be a polynomial of degree $d \geq 2$, let z_0 be a repelling fixed point of p and let L be a linearizer of p at z_0. If R satisfies (1.1) then $\mathrm{A}_R(L)$ is a spider's web if and only if the component of $\mathcal{J}(p)$ which contains z_0 equals $\{z_0\}$.*

Since polynomials for which all critical points converge to ∞ have totally disconnected Julia sets [5, p.85], we obtain, using [16, Theorem 1.4], the following corollary which also implies Rempe's conjecture.

Corollary 1.2. *Let p be a polynomial of degree $d \geq 2$ for which all critical points escape and let L be a linearizer of p. Assume that R satisfies (1.1). Then each of the sets $\mathrm{A}_R(L)$, $\mathrm{A}(L)$ and $\mathrm{I}(L)$ is a spider's web. In particular, this is true whenever $p(z) = z^2 + c$ and c lies outside the Mandelbrot set.*

We believe that the dichotomy established in Theorem 1.1 for the sets $\mathrm{A}_R(L)$ also extends to the sets $\mathrm{A}(L)$ and $\mathrm{I}(L)$. However, we were not able to prove this. For the fast escaping set, such a result would follow if every continuum in $\mathrm{A}(f)$ (or every 'loop') would be contained in some level set $\mathrm{A}_R^l(f)$, which we also believe to be true (compare Question 2 and 3 in [16]).

In the proof of Theorem 1.1, we establish spiders' webs by proving that the corresponding linearizers grow regularly and that there exist simple closed curves arbitrary close to 0 on which the minimum modulus grows fast enough.

Since the order of a linearizer of a quadratic polynomial is given by $\log 2/\log\left|p^{'}(z_0)\right|$, we obtain for any given $\rho \in (0,\infty)$ a linearizer of order ρ whose escaping set is a spider's web.

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2. Preliminaries

The complex plane, the Riemann sphere and the unit disk are denoted by $\mathbb{C}$, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and $\mathbb{D}$, respectively. The circle at 0 with radius r will be denoted by $\mathbb{S}_r$. We write $\mathbb{D}_r(z)$ for the Euclidean disk of radius r centred at z.

If not stated differently, we will assume throughout the article that $f : \mathbb{C} \to \mathbb{C}$ is a non-constant, non-linear entire function; so f is either a polynomial of degree ≥ 2 or a transcendental entire map.

Let $C \subset \mathbb{C}$ be a compact set. The *maximum modulus* $\mathrm{M}(f, C)$ and the *minimum modulus* $\mathrm{m}(f, C)$ of f relative to C are defined to be

$$\mathrm{M}(f, C) := \max_{z\in C} |f(z)| \quad \text{and} \quad \mathrm{m}(f, C) := \min_{z\in C} |f(z)|.$$

In the case when $C = \mathbb{S}_r$ we will simplify the notation by writing $\mathrm{M}(f, r)$ and $\mathrm{m}(f, r)$ for $\mathrm{M}(f, \mathbb{S}_r)$ and $\mathrm{m}(f, \mathbb{S}_r)$, respectively. Finally, recall that the *order* of f is defined as

$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log \mathrm{M}(f, r)}{\log r}.$$

2.1. **Background on dynamics of entire maps.** We denote by $\mathrm{Crit}(f) := \{z \in \mathbb{C} : f'(z) = 0\}$ the set of *critical points*, by $\mathcal{C}(f) := f(\mathrm{Crit}(f))$ the set of *critical values*, and by $\mathcal{A}(f)$ the set of all *(finite) asymptotic values* of f. The elements of $\mathcal{S}(f) = \overline{\mathcal{C}(f) \cup \mathcal{A}(f)}$ are called *singular values* of f, and $\mathcal{S}(f)$ can be characterized as the smallest closed subset of $\mathbb{C}$ such that $f : \mathbb{C} \setminus f^{-1}(\mathcal{S}(f)) \to \mathbb{C} \setminus \mathcal{S}(f)$ is a covering map. If f is a polynomial then $\mathcal{A}(f) = \emptyset$ and $\mathcal{C}(f)$ is finite, so

in this case, $\mathcal{S}(f) = \mathcal{C}(f)$. The *postsingular set* of f is defined to be $\mathcal{P}(f) := \overline{\bigcup_{n\geq 0} f^n(\mathcal{S}(f))}$.

Denote by f^n the n-th iterate of f. A point $w \in \mathbb{C}$ is said to be *exceptional under* f if its backward orbit, i.e., the set of all points z which are mapped to w by some f^n, is finite. The set of all exceptional values of f will be denoted by $\mathcal{E}(f)$. It is well known that $\mathcal{E}(f)$ contains at most one point. We write $\mathcal{O}(f)$ for the set of all (finite) *omitted values* of f.

If z is a periodic point of f of period n, we call $\mu(z) := (f^n)'(z)$ the *multiplier* of z. A periodic point z is called *attracting* if $0 \leq |\mu(z)| < 1$, *indifferent* if $|\mu(z)| = 1$ and *repelling* if $|\mu(z)| > 1$. An attracting periodic point z is called *superattracting* if $\mu(z) = 0$.

The *Fatou set* $\mathcal{F}(f)$ of f is the set of all points that have a neighbourhood in which the iterates of f form a normal family; the *Julia set* $\mathcal{J}(f)$ is defined to be $\mathbb{C} \setminus \mathcal{J}(f)$. For a point $z \in \mathcal{J}(f)$ we denote by $\mathcal{J}_z(f)$ the component of $\mathcal{J}(f)$ that contains z.

2.2. **Poincaré functions.** Let z_0 be a repelling fixed point of f with multiplier λ. By the Kœnigs Linearization Theorem [10, Theorem 8.2], there exists a holomorphic function l defined in a neighbourhood of 0 such that $l(0) = z_0$ and locally, $l^{-1} \circ f \circ l(z) = \lambda z$. It was observed already by Poincaré that l continues to a holomorphic function L on the entire complex plane, meaning there exists an entire map L such that $L(0) = z_0$ and

$$f(L(z)) = L(\lambda z) \tag{2.1}$$

for all $z \in \mathbb{C}$. Every such map is called *linearizer* or *Poincaré function* of f at z_0. A linearizer is unique up to a constant. More precisely, if L satisfies (2.1), then so does $L_c : z \mapsto L(cz)$ for every $c \in \mathbb{C}^*$, and every solution of the equation (2.1) is of this form. If L_1 and L_2 are two linearizers then we say that they have the same *normalization* if $L_1'(0) = L_2'(0)$. We say that L has the *standard normalization* if $L'(0) = 1$.

Proposition 2.1. *Let f_1 and f_2 be entire functions, and assume that there exists a conformal map $\varphi(z) = az + b$ such that*

$$f_2 = \varphi^{-1} \circ f_1 \circ \varphi$$

everywhere in $\mathbb{C}$. If L_1 and L_2 are linearizers of f_1 and f_2 at z_1 and $z_2 = \varphi^{-1}(z_1)$, respectively, with the same normalization, then

$$L_2 = \varphi^{-1} \circ L_1 \circ (\varphi - b),$$

where $(\varphi - b)(z) := \varphi(z) - b = az$.

Proof. Consider the function $\widetilde{L}(z) := \varphi^{-1} \circ L_1 \circ (\varphi - b)(z)$. Then $\widetilde{L}$ satisfies

$$\begin{aligned} f_2 \circ \widetilde{L}(z/\lambda) &= f_2 \circ \varphi^{-1} \circ L_1 \circ (\varphi - b)(z/\lambda) = \varphi^{-1} \circ f_1 \circ L_1(az/\lambda) \\ &= \varphi^{-1} L_1(az) = \widetilde{L}(z). \end{aligned}$$

Since f_1 and f_2 are conformally conjugate, the multipliers at z_1 and z_2 coincide, hence $\widetilde{L}$ is a linearizer of f_2 at $\varphi^{-1}(z_1) = z_2$. Furthermore, $\widetilde{L}'(0) = L_1'(0)$, hence L_1 and $\widetilde{L}$ have the same normalization, yielding $L_2 = \widetilde{L}$. □

In many dynamical settings, conformal conjugacies produce no relevant dynamical consequences, hence it is natural to ask the following: Assume that f_1 and f_2 are as in Proposition 2.1 (so f_1 and f_2 are conformally conjugate entire functions) and let L_1 be a linearizer of f_1. Does there exist a linearizer L_2 of f_2 which is conformally conjugate to L_1 (and hence has the same dynamics)? In general, the answer is no. If namely such a linearizer L_2 would exist, then a corresponding conjugacy, say ψ, would map $\mathcal{S}(L_1)$ bijectively onto $\mathcal{S}(L_2)$, which turns out to be equivalent to the condition

$$\psi(\mathcal{P}(f_1)) = \mathcal{P}(f_2) \tag{2.2}$$

(see Proposition 3.2). Since φ conjugates f_1 and f_2, it already satisfies (2.2), so in particular, the map $\psi^{-1} \circ \varphi$ is a conformal automorphism of $\mathbb{C}$ that fixes the set $\mathcal{P}(f_1)$.

Now if Z is an arbitrary finite subset of $\mathbb{C}$ with at least two elements, then $G_Z := \{h(z) = az + b : a \in \mathbb{C}^*, b \in \mathbb{C}, h(Z) = Z\}$ is a finite group and one can easily check that the map $G_Z \to \mathbb{C}^*, az + b \mapsto a$ is an injective group-homomorphism. Hence G_Z is isomorphic to a finite subgroup of $\mathbb{C}^*$, which must be a cyclic group generated by a root of unity. So every such G_Z is generated by a map of the form $z \mapsto \exp(2\pi ik/n)z + b$ with coprime k and n and $n \leq |Z|$. This allows to phrase necessary geometric conditions on a finite set Z such that G_Z is not trivial. It is clear that such conditions are rather strong; e.g., if $z \mapsto \exp(2\pi ik/n)z + b$ is a generator of G_Z and p its (unique) fixed point in $\mathbb{C}$ then all elements of Z must lie on r circles centred at p, where $r \cdot n \leq |Z \setminus \{p\}|$. To give an explicit dynamical example, one can consider the unique real parameter c, for which $f(z) := z^2 + c$ has a superattracting cycle of period three; one easily sees that $G_{\mathcal{P}(f)}$ is trivial.

However, triviality of $G_{\mathcal{P}(f_1)}$ implies $\psi \equiv \varphi$. So if $\varphi(z) = az + b$, then by Proposition 2.1, every linearizer of f_2 is of the form

$$L_2(z) = \varphi^{-1} \circ L_1 \circ c(\varphi - b)$$

for some $c \in \mathbb{C}^*$, and no such map can be conformally conjugate to L_1 via φ whenever $b \neq 0$ (and $c \neq 1$).

Before the end of this paragraph let us observe that one can iterate f inside the functional equation and obtain

$$f^n \circ L(z) = L \circ \lambda^n(z) \tag{2.3}$$

as an iterated version of (2.1), where λ^n denotes the function $z \mapsto \lambda^n z$.

The growth of the function f and a linearizer L are related in the following sense: If f is transcendental entire then L has infinite order. If f is a polynomial then $\rho(L) = \log d / \log |\lambda|$.

2.3. **Polynomial dynamics near ∞ and repelling fixed points.** If p is a polynomial, the Julia set of p is compact and $\mathrm{I}(p)$ is an open connected subset of $\mathcal{F}(p)$; moreover, it is simply-connected if and only if $\mathcal{J}(p)$ is connected. This property is equivalent to the relation $\mathcal{C}(p) \cap \mathrm{I}(p) = \emptyset$ [10, Lemma 9.4, Theorem 9.5].

Near ∞, the iterates of a polynomial behave in the following simple way.

Proposition 2.2. *Let $p(z) = \sum_{n=0}^{d} a_n z^n$ be a polynomial of degree $d \geq 2$. Then for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for every z with $|z| > R_\varepsilon$, we have*

$$(1-\varepsilon)|a_d| \cdot |z|^d \leq |p(z)| \leq (1+\varepsilon)|a_d| \cdot |z|^d,$$

and $R_\varepsilon \to \infty$ as $\varepsilon \to 0$.

If ε is chosen small enough such that $(1-\varepsilon)|a_d| R_\varepsilon^{d-1} > 1$, then

$$((1-\varepsilon)|a_d|)^{q_n(d)} \cdot |z|^{d^n} \leq |p^n(z)| \leq ((1+\varepsilon)|a_d|)^{q_n(d)} \cdot |z|^{d^n}$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{C}$ with $|z| > R_\varepsilon$, where $q_n(z) := (z^n - 1)/(z - 1) = z^{n-1} + \cdots + z + 1$.

Proof. The first statement is elementary and well-known.

Note that we have chosen ε sufficiently small such that $|z| > R_\varepsilon$ implies $|p(z)| > R_\varepsilon$. We will prove the statement inductivly. So for $n = 1$ we have $q_1(z) = 1$ and the claim follows from the first part. For the iterate $p^{n+1}(z) = p(p^n(z))$ we then obtain

$$\begin{aligned} |p(p^n(z))| &\leq (1+\varepsilon)|a_d||p^n(z)|^d \leq (1+\varepsilon)|a_d| \left[((1+\varepsilon)|a_d|)^{q_n(d)}|z|^{d^n}\right]^d \\ &= ((1+\varepsilon)|a_d|)^{d \cdot q_n(d)+1}|z|^{d^{n+1}} = ((1+\varepsilon)|a_d|)^{q_{n+1}(d)}|z|^{d^{n+1}} \end{aligned}$$

as well as

$$\begin{aligned} |p(p^n(z))| &\geq (1-\varepsilon)|a_d||p^n(z)|^d \geq (1-\varepsilon)|a_d| \left[((1-\varepsilon)|a_d|)^{q_n(d)}|z|^{d^n}\right]^d \\ &= ((1-\varepsilon)|a_d|)^{d \cdot q_n(d)+1}|z|^{d^{n+1}} = ((1-\varepsilon)|a_d|)^{q_{n+1}(d)}|z|^{d^{n+1}}. \end{aligned}$$

□

Near a repelling fixed point of p, we can make the following statement on the escaping set $\mathrm{I}(p)$.

Proposition 2.3. *Let z_0 a repelling fixed point of p. For every $\delta > 0$ there exists a simple closed curve $\gamma_\delta \subset \mathbb{D}_\delta(z_0) \cap \mathrm{I}(p)$ around z_0 if and only if $\mathcal{J}_{z_0}(p) = \{z_0\}$.*

Proof. Let us first assume that for every $\delta > 0$ there exists a simple closed curve $\gamma_\delta \subset \mathrm{I}(p)$ around z_0 such that $\mathrm{dist}(z_0, \gamma_\delta) < \delta$. Then $\mathcal{J}_{z_0}(p)$ is contained in the interior of every γ_δ, hence it must consist of a single point.

If $\mathcal{J}_{z_0}(p) = \{z_0\}$, then for every $\delta > 0$ there exist open, non-empty disjoint sets U_δ and V_δ such that $\mathcal{J}(p) \subset U_\delta \cup V_\delta$, $\mathcal{J}_{z_0}(p) \subset U_\delta$ and $\mathrm{dist}(z_0, U_\delta) < \delta/2$. Furthermore, we can assume U_δ to be connected; otherwise, we replace U_δ by the connected component of U_δ that contains z_0, which is also an open set. By the Plane Separation Theorem [19, Chapter VI, Theorem 3.1], there exists a simple closed curve S_δ which separates z_0 from $\mathcal{J}(p) \cap V_\delta$ such that $S_\delta \cap \mathcal{J}(p) = \emptyset$ and every point in $\mathcal{J}(p) \cap U_\delta$ is at distance less than $\delta/2$ from S_δ. Hence, $\mathrm{dist}(S_\delta, z_0) < \delta$ and $S_\delta \subset \mathcal{F}(p)$. Moreover, the component of $\mathcal{F}(p)$ which contains S_δ must be $\mathrm{I}(p)$ since every bounded component of the Fatou set is simply-connected. □

3. The set of singular values of a linearizer

If not stated differently, we will assume throughout this section that f is an entire function, z_0 a repelling fixed point of f and L a linearizer of f at z_0. We begin with a simple connection between exceptional values of f and omitted values of L.

Proposition 3.1. *The sets $\mathcal{O}(L)$ and $\mathcal{E}(f) \setminus \{z_0\}$ are equal.*

Proof. Since $L(0) = z_0$, the point z_0 is never an omitted value of L. If $a \in \mathbb{C} \setminus \mathcal{E}(f)$, then the backward orbit of a has infinitely many elements. Since L omits at most one finite value, the backward orbit of a under f intersects $L(\mathbb{C})$, i.e., there exists $n \in \mathbb{N}$ and $w \in \mathbb{C}$ with $L(w) \in f^{-n}(a)$. This means $a = f^n(L(w)) = L(\lambda^n w)$, so $a \notin \mathcal{O}(L)$. This proves $\mathcal{O}(L) \subset \mathcal{E}(f) \setminus \{z_0\}$.

Now let $a \in \mathbb{C} \setminus \mathcal{O}(L)$. If $a = z_0$, then we are done. So suppose that $a \neq z_0$. Then there exists $z \neq 0$ with $L(z) = a$. By the iterated functional equation, $L(z/\lambda^j) \in f^{-j}(a)$. Since $z \neq 0$ and L is injective in a neighborhood of 0, the backward orbit of a under f has infinitely many elements. □

Next, we will show that the postsingular set of f and the set of singular values L coincide. This seems to be well-known (and to us, the main parts of the proof have been presented by A. Epstein), but we could not find a reference, which is why we include a proof.

Proposition 3.2. *The following relations are true:*

(*i*) $\mathcal{C}(L) = \bigcup_{n \geq 0} f^n(\mathcal{C}(f)) \setminus \mathcal{E}(f)$.

(ii) $\mathcal{S}(L) = \mathcal{P}(f)$.

Proof. Let $w = L(z) \in \mathcal{C}(L)$, in particular $w \notin \mathcal{O}(L)$. Since $L'(0) \neq 0$, we have $w \neq z_0$. It follows from Proposition 3.1 that $w \notin \mathcal{E}(f)$. Differentiating the iterated functional equation yields

$$0 = (f^n)'(L(z/\lambda^n)) \cdot L'(z/\lambda^n) \cdot \frac{1}{\lambda^n}.$$

Denote by $\mathrm{Crit}(f)$ the set of critical points of f. Since $L'(z/\lambda^n) \neq 0$ if n is large enough, it follows that $L(z/\lambda^n) \in \mathrm{Crit}(f^n)$. Since $\mathrm{Crit}(f^n) = \bigcup_{k=0}^{n-1} f^k(\mathrm{Crit}(f))$ by the chain rule, there exists some $k \leq n-1$ with $L(z/\lambda^n) = f^k(y)$, where $y \in \mathrm{Crit}(f)$. It follows that

$$w = L(z) = f^n(L(z/\lambda^n)) = f^n(f^k(y)) = f^{n+k}(y),$$

i.e., $w \in \bigcup_{n\geq 0} f^n(\mathcal{C}(f))$.

For the other inclusion, let $w \in f^n(\mathcal{C}(f)) \setminus \mathcal{E}(f)$. We want to show that there exists some $z \in L^{-1}(w)$ with $L'(z) = 0$. Again, we differentiate the iterated functional equation and obtain

$$L'(z) = (f^{n+1})'(L(z/\lambda^{n+1})) \cdot L'(z/\lambda^{n+1}) \cdot \frac{1}{\lambda^{n+1}}$$

for all $z \in \mathbb{C}$. There exists some $y \in \mathrm{Crit}(f)$ such that $w = f^{n+1}(y)$. Clearly, $y \notin \mathcal{E}(f)$ since $w \notin \mathcal{E}(f)$. By Proposition 3.1, we have $y \notin \mathcal{O}(L)$, so there exists $z \in \mathbb{C}$ with $y = L(z/\lambda^{n+1})$. It follows by the chain rule that $L'(z) = 0$, and we have $w = f^{n+1}(y) = f^{n+1}(L(z/\lambda^{n+1})) = L(z)$, which finishes the proof of (i).

We now prove (ii). For the composition $f \circ L$ one obtains

$$\mathcal{S}(f \circ L) = S(f|_{f(\mathbb{C})}) \cup \overline{f(\mathcal{S}(L))} = \mathcal{S}(f) \cup f(\mathcal{S}(L)),$$

since every Picard value of f is also a singular value of f. Let us abbreviate $S := \mathcal{S}(f) \cup f(\mathcal{S}(L))$. Since the composition

$$\mathbb{C} \setminus L^{-1}(f^{-1}(S)) \xrightarrow{L} \mathbb{C} \setminus f^{-1}(S) \xrightarrow{f} \mathbb{C} \setminus S$$

is a covering map, it follows from (2.1) that

$$\mathbb{C} \setminus \lambda^{-1} \cdot L^{-1}(f^{-1}(S)) \xrightarrow{\lambda} \mathbb{C} \setminus L^{-1}(S) \xrightarrow{L} \mathbb{C} \setminus S$$

must be a covering map as well. Hence

$$\mathcal{S}(f) \cup f(\mathcal{S}(L)) = S \supset \mathcal{S}(L \circ \lambda) = \mathcal{S}(L).$$

The argument is commutative with respect to (2.1), so we obtain the opposite inclusion, yielding the equality $\mathcal{S}(L) = \mathcal{S}(f) \cup f(\mathcal{S}(L))$. But for a point $w \in \mathcal{S}(f)$, this implies that $w \in \mathcal{S}(L)$, and so $f(w) \in f(\mathcal{S}(L)) \subset \mathcal{S}(L)$. By proceeding inductively, it follows for every $n \in \mathbb{N}$ that $f^n(w) \in \mathcal{S}(L)$, hence $\mathcal{P}(f) \subset \mathcal{S}(L)$.

Let $w \in \mathbb{C} \setminus \mathcal{P}(f)$. Then there exists a disk $D \ni w$ such that all inverse branches of all iterates of f exist in D. Let $v \in D$ and $z \in L^{-1}(v)$, and define $z_n := z/\lambda^n$ and $v_n := L(z_n)$. Let g_n be the branch of $(f^n)^{-1}$

such that $g_n(v) = v_n$ and let $D_n := g_n(D)$. By the Shrinking Lemma in [8], it follows that the diameter of the domains D_n converges to 0 (Actually, the statement in [8] is not phrased such that it completely covers our setting but the proof gives what we require). We choose a domain U in which L is injective. Then for n large enough, D_n lies in $L(U)$. Let T be the branch of L^{-1} that maps D_n into U. Then we have

$$L \circ (\lambda^n \circ T \circ g_n)(z) = f^n \circ L \circ \underbrace{(T \circ g_n)(z)}_{\in U} = (f^n \circ g_n)(z) = z.$$

Since z is an arbitrarily chosen preimage of an arbitrary point in D, all inverse branches of L can be defined in D. Hence $w \in \mathbb{C} \setminus \mathcal{S}(L)$. □

If f is a polynomial then $\mathcal{A}(L)$ is contained in the union of attracting and parabolic periodic cycles and the accumulation points of recurrent critical points in $\mathcal{J}(f)$ [3, Theorem 1]. Depending on the location of the repelling fixed point z_0 relative to $\mathcal{F}(f)$, we can exclude certain attracting cycles of f as asymptotic values for L.

Proposition 3.3. *Let f be a polynomial and let $w \in \mathcal{A}(L)$. If w is an attracting periodic point of f, then z_0 lies in the boundary of the immediate attracting basin of w.*

Proof. Let w be an attracting periodic point of f of period k and assume that w is an asymptotic value of L. Then there exists a path γ to ∞ for which $\lim_{t\to\infty} L(\gamma(t)) = w$. Since $w \in \mathcal{F}(f)$ and $\mathcal{F}(f)$ is open, we can assume that $L(\gamma) \subset \mathcal{F}(f)$. It follows from (2.3) that every path $\gamma_n(t) := \lambda^{-n} \cdot \gamma(t)$ is again an asymptotic path for L. Moreover, the limit of L along γ_{nk} is contained in $f^{-nk}(w)$. On the other hand, every such limit point must lie in the set of attracting periodic points [3, Theorem 1], hence it follows that $\lim_{t\to\infty} L(\gamma_{nk}(t)) = w$. Furthermore, for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$, the curve $L(\gamma_{nk})$ intersects $\mathbb{D}_\varepsilon(z_0)$. Hence $z_0 \in \partial A^*(w)$. □

Recall that a point $z \in \mathcal{J}(f)$ is called a *buried point* if it does not belong to the boundary of any Fatou component (other that $\mathrm{I}(f)$).

Corollary 3.4. *If f is a polynomial and z_0 is a buried point (of f) then L has no asymptotic values.*

Linearizers can be very useful to construct entire or meromorphic functions whose set of singular values satisfies certain conditions. For instance, in [9], there was given an example of an entire function of finite order with no asymptotic values and only finitely many critical values such that the ramification degree on its Julia set was unbounded; the constructed function was a linearizer of a certain hyperbolic quadratic polynomial. Here we want to show another interesting example that can be constructed using linearizers, in this case of a transcendental entire function f.

Let $f(z) := \mu \exp(z)$ where $\mu \in \mathbb{C}$ is chosen such that $\bigcup_{n\geq 0} f^n(0)$ is dense in $\mathbb{C}$. The existence of such parameters is well-known. By [7, Theorem 2], the function f has infinitely many fixed points. Since $\mathcal{S}(f) = \{0\}$, at most one of them is non-repelling [1, Theorem 7], so we can pick a repelling fixed point z_0 of f. Let L be a linearizer of f at z_0. It follows from the functional equation that 0 is an omitted value of L. By Proposition 3.2, every point $w_n := f^n(0)$ is an asymptotic value of L. It is also not hard to check that L has a direct singularity lying over each of the points w_n. (For a clarification of terminology, see e.g. [3]; our last claim also follows from [3, Theorem 1.4], which is formulated for linearizers of rational maps only, but extends to linearizers of transcendental entire maps with the same proof.) Hence L is a map for which the set of projections of direct singularities (or direct asymptotic values) is dense in $\mathbb{C}$. This is optimal, since by a theorem of Heins [6], the set of projections of direct singularities is always countable.

4. Maximum and minimum modulus estimates

In the remaining part of the article we prove Theorem 1.1. From now on, we consider an arbitrary but fixed polynomial p of degree $d \geq 2$, hence p can be written as

$$p(z) = \sum_{i=0}^{d} a_i z^i = a_0 + a_1 z \cdots + a_d z^d, \quad a_d \neq 0.$$

For every $\varepsilon > 0$ we pick a constant $R_\varepsilon \geq 1$ for which the conclusion of Proposition 2.2 is satisfied, and such that $\varepsilon_1 < \varepsilon_2$ implies $R_{\varepsilon_1} > R_{\varepsilon_2}$. We assume that p has a repelling fixed point z_0 with multiplier λ, and we denote by L a linearizer of p at z_0. We also pick a constant $R_L \geq 1$ such that $\mathrm{M}(L, s) > s$ for all $s \geq R_L$.

Lemma 4.1 (Regularity of growth)**.** *Let $\varepsilon > 0$, $r > \max\{R_\varepsilon, R_L\}$ and define $k_\varepsilon := \log((1-\varepsilon)|a_d|)$ and $K_\varepsilon := \log((1+\varepsilon)|a_d|)$. Then*

$$\prod_{i=0}^{n-1}\left(d + \frac{\log k_\varepsilon}{\log \mathrm{M}(L, |\lambda|^i r)}\right) \leq \frac{\log \mathrm{M}(L, |\lambda|^n r)}{\log \mathrm{M}(L, r)} \leq \prod_{i=0}^{n-1}\left(d + \frac{\log K_\varepsilon}{\log \mathrm{M}(L, |\lambda|^i r)}\right)$$

holds for all $n \in \mathbb{N}$.

Proof. Let r be as assumed, and let $\tilde{z} \in \mathbb{S}_r$ be a point for which $L(\tilde{z}) \geq L(z)$ for all $z \in \mathbb{S}_r$. Let $\tilde{w} := L(\tilde{z})$. Then $|\tilde{w}| = \mathrm{M}(L, r)$ and it follows from the functional equation (2.1) and Proposition 2.2 that

$$\begin{aligned} \log \mathrm{M}(L, |\lambda| r) &= \log \mathrm{M}(p \circ L, r) = \log \mathrm{M}(p, L(\mathbb{S}_r)) \geq \log p(\tilde{w}) \\ &\geq \log\big((1-\varepsilon)|a_d| \cdot |\tilde{w}|^d\big) = k_\varepsilon + d \cdot \log \mathrm{M}(L, r), \end{aligned}$$

and

$$\begin{aligned} \log \mathrm{M}(L, |\lambda| r) &= \log \mathrm{M}(p, L(\mathbb{S}_r)) \leq \log \mathrm{M}(p, \mathrm{M}(L, r)) \\ &\leq \log\big((1+\varepsilon)|a_d| \cdot \mathrm{M}(L, r)^d\big) = K_\varepsilon + d \cdot \log \mathrm{M}(L, r). \end{aligned}$$

Hence,

$$\left(\frac{k_\varepsilon}{\log \mathrm{M}(L,r)} + d\right) \leq \frac{\log \mathrm{M}(L,|\lambda| r)}{\log \mathrm{M}(L,r)} \leq \left(\frac{K_\varepsilon}{\log \mathrm{M}(L,r)} + d\right).$$

The statement now follows immediately from the fact that

$$\frac{\log \mathrm{M}(L,|\lambda|^n r)}{\log \mathrm{M}(L,r)} = \frac{\log \mathrm{M}(L,|\lambda|^n r)}{\log \mathrm{M}(L,|\lambda|^{n-1} r)} \cdot \ \ldots \ \cdot \frac{\log \mathrm{M}(L,|\lambda| r)}{\log \mathrm{M}(L,r)}.$$

□

Lemma 4.2. *For every $k \in \mathbb{N}$ there exists $R_k > 0$ such that for all $R > R_k$, $m \leq d^k$ and $n > k$,*

$$\mathrm{M}(L, r_n) > r_{n+1}^m,$$

where the sequence (r_n) is defined by

$$r_n := |\lambda|^n \cdot \mathrm{M}^n(L,R).$$

Moreover, we can choose $R_1 = 2 \cdot \max\left\{ \log|a_d|, \log\frac{2}{|a_d|}, \log|\lambda| \right\}$.

Proof. Let $\varepsilon \in (0,1/2)$ be arbitrary but fixed, and let $R > \max\{R_L, R_\varepsilon\}$. It follows from Lemma 4.1 with $r = \mathrm{M}^n(L,R)$ that

$$\begin{aligned}
\log \mathrm{M}(L,r_n) &= \log \mathrm{M}(L, |\lambda|^n \, \mathrm{M}^n(L,R)) \\
&\geq \prod_{i=0}^{n-1}\left(d + \frac{\log k_\varepsilon}{\log \mathrm{M}(L, |\lambda|^i \, \mathrm{M}^n(L,R))}\right) \cdot \log \mathrm{M}(L, \mathrm{M}^n(L,R)) \\
&\geq \left(d - \frac{|\log k_\varepsilon|}{\log R}\right)^n \cdot \log \mathrm{M}^{n+1}(L,R).
\end{aligned}$$

By definition,

$$\begin{aligned}
\log r_{n+1}^m &= m \log(|\lambda|^{n+1} \, \mathrm{M}^{n+1}(L,R)) \\
&= m(n+1)\log|\lambda| + m \log \mathrm{M}^{n+1}(L,R).
\end{aligned}$$

Define $c_R := \frac{|\log k_\varepsilon|}{\log R}$. We want to show that there exists R_k such that when $R > R_k$, $m \leq d^k$ and $n \geq k+1$, then

$$\log \mathrm{M}^{n+1}(L,R) \cdot ((d - c_R)^n - m) > m(n+1)\log|\lambda|.$$

Obviously, it is sufficient if the wanted constant R_k satisfies

$$\log R_k \cdot ((d - c_{R_k})^n - d^k) > d^k (n+1) \log|\lambda|$$

for all $n \geq k+1$, and this is certainly true when we choose R_k sufficiently large. We will omit the details since they follow from elementary calculus; however, one can prove inductively that every R_k with $\log R_k > \max\{2|\log k_\varepsilon|, \frac{2k}{d}|\log k_\varepsilon|, \frac{\sqrt{\mathrm{e}}\log|\lambda|}{(2-\sqrt{\mathrm{e}})(k+2)}\}$ is sufficiently large. Hence for $k = 1$ we can choose $R_1 = 2\max\{|\log|a_d||, |\log\frac{1}{2}|a_d||, \log|\lambda|)$ since $|\log k_\varepsilon| = |\log((1-\varepsilon)|a_d|)|$ and $\varepsilon \in (0,1/2)$, and since $\frac{\sqrt{\mathrm{e}}}{3\cdot(2-\sqrt{\mathrm{e}})} < 2$. □

Lemma 4.3 (growth of minimum modulus). *Suppose that $\mathcal{J}_{z_0}(p) = \{z_0\}$ and let $m \in \mathbb{N}_{>1}$. Then there exists $R_m > 0$ with the following property: For every $r > R_m$ there is a simple closed curve Γ^r separating $\mathbb{S}_r$ and $\mathbb{S}_{r^m}$ such that*

$$\mathrm{m}(L, \Gamma^r) > \mathrm{M}(L, r).$$

Proof. Let D be a disk around 0 such that $L|_D$ is conformal. Let $\delta > 0$ be sufficiently small such that $\mathbb{D}_\delta(z_0) \subset L(D)$. By Proposition 2.3 there exists a simple closed curve $\gamma_\delta \subset \mathbb{D}_\delta(z_0) \cap \mathrm{I}(p)$ which surrounds z_0. Since such a curve exists in the intersection of every arbitrarily small neighbourhood of z_0 and $\mathrm{I}(p)$, we can assume w.l.o.g. that $D = \mathbb{D}$. Let $\Gamma_\delta = L^{-1}(\gamma_\delta) \cap \mathbb{D}$. Then Γ_δ is a simple closed curve surrounding 0. Define

$$s := \min_{z \in \Gamma_\delta} |z| = \mathrm{dist}(0, \Gamma_\delta) \quad \text{and} \quad t := \max_{z \in \Gamma_\delta} |z|.$$

Obviously, both s and t are finite and positive constants.

Let $r > \left(\frac{|\lambda| \cdot t}{s}\right)^{\frac{1}{m-1}}$ be an arbitrary but fixed number. We define $l(r)$ to be the unique integer for which

$$|\lambda|^{l(r)-1} \leq r < |\lambda|^{l(r)}.$$

Similarly, for the external radius t of the curve Γ_δ we denote by $l(t)$ the unique natural number for which

$$t \cdot |\lambda|^{l(t)} \leq r^m < t \cdot |\lambda|^{l(t)+1}.$$

(Note that the lower bound for r implies that $s \cdot |\lambda|^{l(t)} > r$.) By taking logarithms we obtain the equivalent equations

$$l(r) - 1 \leq \frac{\log r}{\log |\lambda|} < l(r)$$

and

$$l(t) \leq \frac{m \cdot \log r - \log t}{\log |\lambda|} < l(t) + 1.$$

A combination of these two inequalities yields

$$m \cdot l(r) - \left(\frac{\log t}{\log |\lambda|} + m + 1\right) < l(t) < m \cdot l(r) - \frac{\log t}{\log |\lambda|}. \tag{4.1}$$

Let us fix an $\varepsilon \in (0, 1/2)$. Let $j \in \mathbb{N}$ be minimal with the property that $p^j(\gamma_\delta) \subset \{z : |z| > R_\varepsilon\}$. Note that there is a unique integer j with this property since γ_δ is a compact subset of $\mathrm{I}(p)$. We define

$$\Gamma^r := \{z \in \mathbb{C} : \lambda^{-l(t)} \cdot z \in \Gamma_\delta\}.$$

Observe that Γ^r separates $\mathbb{S}_r$ and $\mathbb{S}_{r^m}$. In order to simplify the calculations, let us consider the logarithms of the minimum and maximum

modulus. Using Proposition 2.2, these can be estimated in the following way:

$$\begin{aligned}\log \mathrm{m}(L,\Gamma^r) &= \log \mathrm{m}(p^{l(t)}\circ L,\Gamma_\delta) = \log \mathrm{m}(p^{l(t)},\gamma_\delta) \geq \log \mathrm{m}(p^{l(t)-j},R_\varepsilon)\\ &\geq \log\{((1-\varepsilon)|a_d|)^{q_{l(t)-j}(d)}\cdot R_\varepsilon^{d^{l(t)-j}}\}\\ &= q_{l(t)-j}(d)\cdot\log((1-\varepsilon)|a_d|) + d^{l(t)-j}\cdot\log R_\varepsilon,\end{aligned}$$

$$\begin{aligned}\log \mathrm{M}(L,r) &= \log \mathrm{M}(p^{l(r)},L(\mathbb{S}_{r\cdot|\lambda|^{-l(r)}})) \leq \log \mathrm{M}(p^{l(r)},R_\varepsilon)\\ &\leq \log\{((1+\varepsilon)|a_d|)^{q_{l(r)}(d)}\cdot R_\varepsilon^{d^{l(r)}}\}\\ &\leq q_{l(r)}(d)\cdot\log((1+\varepsilon)|a_d|) + d^{l(r)}\cdot\log R_\varepsilon.\end{aligned}$$

Equation (4.1) yields the relation $m{\cdot}l(r)-C < l(t) < m{\cdot}l(r)+c$ with the constants $C := \log t/\log|\lambda|+m+1$ and $c := \log t/\log|\lambda|$. Furthermore, by Proposition 2.2 we can estimate the polynomials $q_{n+1}(d) = d^n+\ldots+d+1 = (d^{n+1}-1)/(d-1)$ by $d^n \leq q_{n+1}(d) \leq d^{n+1}$. Together, we obtain

$$\begin{aligned}\log \mathrm{m}(L,\Gamma^r) &> d^{m\cdot l(r)-C-j-1}\cdot\log((1-\varepsilon)|a_d|) + d^{m\cdot l(r)-C-j}\cdot\log R_\varepsilon\\ &= d^{m\cdot l(r)}\cdot\frac{\log((1-\varepsilon)|a_d|R_\varepsilon^d)}{d^{C+j+1}},\end{aligned}$$

$$\begin{aligned}\log \mathrm{M}(L,r) &\leq d^{l(r)}\cdot\log((1+\varepsilon)|a_d|) + d^{l(r)}\cdot\log R_\varepsilon\\ &= d^{l(r)}\cdot\log((1+\varepsilon)|a_d|R_\varepsilon)\end{aligned}$$

as new lower and upper bounds for the minimum and maximum modulus, respectively. Since $m \geq 2$, it is sufficient to find a constant R_m such that for all $r > R_m$,

$$\begin{aligned}& d^{2l(r)}\cdot\frac{\log((1-\varepsilon)|a_d|R_\varepsilon^d)}{d^{C+j+1}} > d^{l(r)}\cdot\log((1+\varepsilon)|a_d|R_\varepsilon)\\ \Longleftrightarrow\quad & d^{l(r)} > \frac{\log((1+\varepsilon)|a_d|R_\varepsilon)}{\log((1-\varepsilon)|a_d|R_\varepsilon^d)}\cdot d^{C+j+1} =: l_\varepsilon.\end{aligned}$$

Hence $R_m := \max\left\{\left(\frac{|\lambda|\cdot t}{s}\right)^{\frac{1}{m-1}}, |\lambda|^{\frac{\log l_\varepsilon}{\log d}}\right\}$ is sufficiently large. □

Proof of Theorem 1.1. Let us start with the case when $\mathcal{J}_{z_0}(p) \neq \{z_0\}$. Assume that $A_R(L)$ is a spider's web for some sufficiently large R. By definition, there exists a sequence of bounded simply-connected domains G_n such that $G_n \subset G_{n+1}$, $\partial G_n \subset A_R(L)$ for $n \in \mathbb{N}$, and $\bigcup G_n = \mathbb{C}$. We can assume w.l.o.g. that every G_n contains 0 (since this is true anyway for all sufficiently large n).

By Proposition 2.3, for every $n \in \mathbb{N}$, the curve $L(\partial G_n)$ intersects the filled Julia set of p. Let $K > 0$ be the radius of the smallest disk around 0 which contains the (filled) Julia set of p. Then there exists a sequence of points $w_n \in \partial G_n$ such that $|L(w_n)| \leq K$. But this contradicts the assumption that all points $z \in \partial G_n$ satisfy $|L(z)| \geq \mathrm{M}(L,R)$.

Let us now consider the situation when $\mathcal{J}_{z_0}(p) = \{z_0\}$. By [16, Theorem 8.1] it is sufficient to find a sequence of bounded simply-connected domains G_n such that for all (sufficiently large) n,

$$G_n \supset \{z \in \mathbb{C} : |z| < \mathrm{M}^n(L, R)\}$$

and

$$G_{n+1} \text{ is contained in a bounded component of } \mathbb{C} \setminus L(\partial G_n).$$

Let R_1 be the constant from Lemma 4.2, and set $R := \max\{R_L, R_1\}$. For $n \in \mathbb{N}$ let $r_n := |\lambda|^n \, \mathrm{M}^n(L, R)$ (see also Lemma 4.2). By Lemma 4.3, there exists a simple closed curve Γ^{r_n} separating $\mathbb{S}_{r_n}$ and $\mathbb{S}_{r_n^d}$ such that $\mathrm{m}(L, \Gamma^{r_n}) > \mathrm{M}(L, r_n)$. We define G_n to be the interior of Γ^{r_n}. Then every G_n is a bounded simply-connected domain with

$$G_n \supset \{z \in \mathbb{C} : |z| < r_n\} \supset \{z \in \mathbb{C} : |z| < \mathrm{M}^n(L, R)\}.$$

Furthermore, it follows from Lemma 4.2 with $m = d$ that

$$\mathrm{m}(L, \partial G_n) = \mathrm{m}(L, \Gamma^{r_n}) > \mathrm{M}(L, r_n) > r_{n+1}^d > \max_{z \in \partial G_{n+1}} |z|,$$

hence G_{n+1} is contained in a bounded component of $\mathbb{C} \setminus L(\partial G_n)$ and the claim follows. $\square$

Note that Corollary 1.2 is an immediate consequence of Theorem 1.1.

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Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, 24118 Kiel, Germany
E-mail address: helenam@math.uni-kiel.de

Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, 24118 Kiel, Germany
E-mail address: peter@math.uni-kiel.de